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Incoming and Outgoing Waves*

C. L. EPSTEIN

Department of Mathematics, Princeton University, Princeton, New Jersey 08544

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1. INTRODUCTION

In this paper, we study spaces of solutions to the wave equation

$$u_{tt} + Lu = 0, \tag{1.1}$$

where

$$Lf = -f_{xx} + qf,$$

which arise in Lax–Phillips scattering theory. For technical simplicity we will work on the half line, $[0, \infty]$.

The natural Hilbert space is data of finite energy; energy is defined with respect to the operator L by

$$E_L(u, u_t) = (Lu, u) + (u_t, u_t).$$

If $u(0, t) = 0$ the energy can be written more symmetrically as

$$E_L(u, u_t) = \int_0^\infty (u_x^2 + qu^2 + u_t^2)\, dx.$$

In order that E_L define a norm it is necessary that

$$E_L(u, u_t) > 0 \qquad \text{for} \quad (u, u_t) \in C_0^\infty \oplus C_0^\infty. \tag{1.2}$$

To have good control on the closure of $C_0^\infty \oplus C_0^\infty$ in the E_L norm we make the following assumptions on q:

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(1) $q \geqslant M > -\infty$.

(2) For every $x > 0$ there is a constant $\infty > C_x > 0$ such that if $u \in C_0^\infty(\mathbb{R}^+)$ then

$$\int_0^x u^2 dx \leqslant C_x \int_0^\infty u_x^2 + qu^2. \tag{1.3}$$

For a smooth potential which decays at ∞ (1.3) follows if the solution to

$$L\phi = 0$$
$$\phi(0) = 0$$

does not vanish for $x > 0$ and behaves as $x \to \infty$ like cx for a $c \neq 0$. For a potential satisfying (1) and (2) we define $\mathscr{H}_L$ to be the closure of $C_0^\infty \oplus C_0^\infty$ in the E_L-norm.

More generally we can study local energy:

$$E_{L,X}(u, u_t) = \int_0^X (u_x^2 + qu^2 + u_t^2)\, dx.$$

For smooth initial data one easily proves

$$\int_0^x [u_x^2(x, 0) + qu^2(x, 0) + u_t^2(x, 0)]\, dx$$
$$\geqslant \int_0^{x-t} [u_x^2(x, t) + qu^2(x, t) + u_t^2(x, t)]\, dx. \tag{1.4}$$

For a potential satisfying (1.3) we obtain (1.4) for all data in $\mathscr{H}_L$ by taking Cauchy sequences of smooth data. The physical interpretation of (1.4) is the finite propagation speed of the wave equation.

In light of (1.4) it is natural to expand our considerations to data of locally finite energy:

Let $\mathscr{H}_L^{\mathrm{loc}}$ consist of all data (f, g) such that $f \in W_{\mathrm{loc}}^{1,2}$, $g \in L_{\mathrm{loc}}^2$ and

$$\int_0^X [f_x^2 + qf^2 + g^2]\, dx < \infty \qquad \text{for every} \quad X > 0 \quad f(0) = 0. \tag{1.5}$$

It follows from (1.4) that data initially of locally finite energy remain of locally finite energy. In most of what follows we will assume that the potential satisfies (1) and (2) stated above.

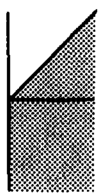
In Lax–Phillips scattering theory one distinguishes two subspaces of data: an incoming and an outgoing subspace. We define these as follows:

The incoming subspace, $D_-(L)$, consists of data in $\mathscr{H}_L$ such that the corresponding solution to (1.1) vanishes in the set

$$\{(x, t): t<0, x< -t\}.$$

The outgoing subspace, $D_+(L)$ consists of the data in $\mathscr{H}_L$ such that the corresponding solution to (1.1) vanishes in the set

$$\{x, t): t>0, x<t\}.$$



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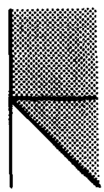

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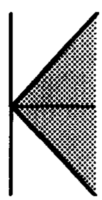
When $q \neq 0$ it is not clear, a priori, that $D_\pm(L)$ have anything in them. Using the methods in this paper one can show that for large classes of potentials $\mathscr{H}_L = D_+(L) + D_-(L)$. We will not pursue this question but instead study the intersection of $D_+(L)$ and $D_-(L)$. A solution with data in $D_+(L) \cap D_-(L)$ vanishes in the set

$$\{(x, t): x<|t|\}.$$



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We will call such a solution an IO-solution. The existence of IO-solutions of locally finite energy is intimately tied to the well posedness of the problem

$$\begin{aligned} Lu &= g \\ u(0) &= 0 \\ u'(0) &= 1. \end{aligned} \tag{1.6}$$

In Section 2 we will show that for many potentials for which (1.6) is well posed there are no IO-solutions of locally finite energy. In Section 4 we will show that there are IO-solutions for potentials with a $1/x^2$ singularity at zero. For such potentials (1.6) is not well posed. Finally, we will show that if q satisfies

$$\int_0^\infty x \left| q - \frac{l(l+1)}{\sinh^2 x} \right| dx < \infty \tag{1.7}$$

for an $l > 1/2$, then $D_+(L) \cap D_-(L)$ is not empty.

A potential satisfying (1.7) defines a "small" perturbation of the operator

$$L_l = -\frac{d^2}{dx^2} + \frac{l(l+1)}{\sinh^2 x}.$$

This operator arises in the study of the Laplace–Beltrami operator on hyperbolic space forms. In the course of their study of the spectral theory for the Laplacian in $\mathbb{H}^n$ [2], Lax and Phillips explicitly constructed finite energy IO-solutions to

$$u_{tt} + L_l u = 0, \qquad l = 1, 2, 3, ...$$

Their work was the main source of inspiration for our study.

The results of the first four sections are applied to study wave equations

$$u_{tt} = (\Delta - q(r))\, u$$

in hyperbolic and Euclidean space. We show that the existence or non-existence of finite energy incoming and outgoing solutions is a scattering invariant.

Before proceeding to the body of the paper we would like to define the notion of a weak solution to (1.1):

$u(x, t)$ is a weak solution to (1.1) with distributional initial data (f, g) if

$$\int_0^T \int_0^\infty u(x, t)(\phi_{tt} + L\phi)\, dx\, dt = \langle \phi_t(x, 0), f \rangle - \langle \phi(x, 0), g \rangle. \tag{1.8}$$

For every $\phi \in C_0^\infty((\mathbb{R}^+ \times \mathbb{R})$ such that

(i) $\phi(x, T) = \phi_t(x, T) = 0$,

(ii) $\phi(0, t) = 0$,

solutions with data in $\mathscr{H}_L^{\text{loc}}$ satisfy (1.8) with $(f, g) \in W_{\text{loc}}^{1,2} \oplus L_{\text{loc}}^2$.

2. Non-singular Potentials

In this section we will show that if q satisfies the following hypotheses:

(1) $\int_0^1 x|q(x)|\,dx < \infty$.

(2) For every positive function $\phi \in C_0^\infty(\bar{\mathbb{R}}_+)$ the operator

$$L\phi = -\frac{d^2}{dx^2} + \phi q \tag{2.1}$$

is essentially self adjoint on $C_0^\infty(\mathbb{R}_+)$.

(3) q is smooth away from the origin and satisfies the positivity conditions presented in the Introduction, (1.3).

then there is no IO-solution to (1.1) (or (1.8)) of locally finite energy.

Observe that if two potentials q_1 and q_2 agree on $[0, X]$ and two sets of data (f^1, g^1), (f^2, g^2) agree on $[0, X]$ then the solutions to the Cauchy problems

$$\begin{gathered} u^1_{tt} = u^1_{xx} - q_1 u^1 \\ u^1(x, 0) = f^1(x); \qquad u^1_t(x, 0) = g^1(x); \qquad u^1(0, t) = 0 \\ u^2_{tt} = u^2_{xx} - q_2 u^2 \\ u^2(x, 0) = f^2(x); \qquad u^2_t(x, 0) = g^2(x); \qquad u^2(0, t) = 0 \end{gathered}$$

satisfy

$$u^1(x, t) = u^2(x, t), \qquad x \leqslant X - |t|.$$

It is for this reason that we do not require any asymptotic behavior for q near ∞. Whenever we are addressing a question which is local in space and time it suffices to consider compactly supported potentials and initial data.

Now we will prove two technical lemmas on weak solutions to (1.1).

Lemma 2.1. *Suppose q satisfies (2.1)(2) and (3) and that $u(x, t)$ is a weak solution to (1.1) with locally finite energy. If $\Psi(t) \in C_0^\infty(\mathbb{R})$ then*

$$u_\Psi(x) = \int_0^\infty \Psi(t)\, u(x, t)\, dt$$

is well defined and is in $C^\infty(\mathbb{R}_+)$.

Proof. By the remarks above we can assume that $q(x)$ and the initial data for $u(x, t)$ are compactly supported. Let $\bar{L}$ denote the self adjoint extension of L on $C_0^\infty(\mathbb{R}_+)$. Let E_λ denote the resolution of the identity defined by $\bar{L}$.

A weak solution in $\mathscr{H}_L^{\text{loc}}$ satisfies the following inequality:

$$\int_0^x [u_x^2(x,t)+qu^2(x,t)+u_t^2(x,t)]\,dx$$
$$\leqslant \int_0^{x+t} [u_x^2(x,0)+qu^2(x,0)+u_t^2(x,0)]\,dx. \tag{2.2}$$

If we integrate (2.2) in time we obtain

$$(2T)^{-1}\int_{-T}^{T}\int_0^x (u_x^2+qu^2+u_t^2)\,dx\,dt$$
$$\leqslant \int_0^{x+T} [u_x^2(x,0)+u_t^2(x,0)+qu^2(x,0)]\,dx. \tag{2.3}$$

From (2.3) and the positivity hypothesis on q it follows that $u(x,t)\in W^{1,2}_{\text{loc}}(\mathbb{R}_+\times\mathbb{R})$. The restrictions of $u(x,t)$ to vertical and horizontal lines have locally uniformly bounded L^2_{loc} norms. Thus the integral defining $u_\Psi(x)$ is locally uniformly convergent.

To show that $u_\Psi(x)$ is smooth we use the spectral representation of $\bar{L}$. One easily checks that if $u(x,0)=f(x)$; $u_t(x,0)=g(x)$ lie in L^2 then:

$$u(x,t)=\int \cos\sqrt{\lambda t}\,dE_\lambda f+\frac{\sin\sqrt{\lambda}\,t}{\sqrt{\lambda}}\,dE_\lambda g.$$

Therefore

$$u_\Psi(x)=\int_{-\infty}^{\infty}\left[\int \cos\sqrt{\lambda t}\,\Psi(t)\,dE_\lambda f+\frac{\sin\sqrt{\lambda t}}{\sqrt{\lambda}}\,dE_\lambda g\right]dt.$$

We can interchange the order of the integrations to obtain

$$u_\Psi(x)=\int \hat{\Psi}_e(\sqrt{\lambda})\,dE_\lambda f+\frac{\hat{\Psi}_0(\sqrt{\lambda})}{\sqrt{\lambda}}\,dE_\lambda g$$

with

$$\hat{\Psi}_e(k)=\frac{1}{2}\int_0^\infty [\Psi(t)+\Psi(-t)]\,e^{-ikt}dt$$
$$\hat{\Psi}_0(k)=\frac{1}{2i}\int_0^\infty [\Psi(t)-\Psi(-t)]\,e^{-ikt}dt.$$

Since $\Psi \in C_0^\infty$ $\hat{\Psi}_e(\sqrt{\lambda})$ and $\hat{\Psi}_0(\sqrt{\lambda})/\sqrt{\lambda}$ are rapidly decreasing and thus $u_\Psi(x)$ is the sum of two functions in the domain of $(\bar{L})^k$ for every $k>0$. L^k is a positive elliptic operator of order $2k$. The local regularity theory for such operators implies that

$$u_\Psi \in H^{2k}_{\text{loc}}(\mathbb{R}_+) \qquad \text{for every} \quad k>0.$$

The lemma follows from the Sobolev embedding theorem. ∎

Though we do not use the following lemma until Section 4, it fits naturally with Lemma 2.1, so we include it here:

LEMMA 2.2. *Suppose $q(x)$ satisfies* (2.1)(2) *and* (3) *and $u(x,t)$ is a weak solution to* (1.1) *with distribution initial data (f, g). Furthermore suppose that for some $T>0$*

$$\int_0^T \int_0^{X+T} [u_x^2(x,t) + qu^2(x,t) + u_t^2(x,t)]\,dx\,dt < \infty.$$

Then $(f(x), g(x)) \in \mathscr{H}_L^{\text{loc}}$ on $[0, X+T]$, moreover

$$\int_0^T \int_0^{X+t} [u_x^2 + u_t^2 + qu^2]\,dx\,dt$$

$$\geqslant s\,T \int_0^{X+T-s} [u_x^2(x,0) + u_t^2(x,0) + qu^2(x,0)]\,dx. \tag{2.4}$$

Proof. If $u(x,t)$ is a smooth solution with locally finite energy vanishing on $x=0$ then the estimate

$$\int_0^{X+t} [u_x^2(x,t) + qu^2(x,t) + u_t^2(x,t)]\,dx$$

$$\geqslant \int_0^X [u_x^2(x,0) + qu^2(x,0) + u_t^2(x,0)]\,dx \tag{2.5}$$

is elementary. If we integrate (2.5) in time we obtain (2.4) for smooth data. Again we can extend (2.5) to data with locally finite energy by taking $\mathscr{H}_L^{\text{loc}}$ Cauchy sequences of smooth data. To obtain convergence of the left-hand side of (2.4) from convergence of initial data in $\mathscr{H}_L^{\text{loc}}$ we employ (2.3).

The hypotheses on q imply that a solution satisfying the hypotheses of the lemma is in $W^{1,2}_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R})$ and thus has L^2_{loc}-bounded traces on horizontal and vertical lines. If $\Psi \in C_0^\infty$ then

$$u_\Psi(x,t) = \int u(x,s)\,\Psi(t-s)\,ds$$

is easily seen to be a weak solution with initial data

$$u_\Psi(x, 0) = \int u(x, s)\, \Psi(-s)\, ds$$

$$\frac{\partial u_\Psi}{\partial t}(x, 0) = \int u(x, s)\, \Psi'(-s)\, ds.$$

In particular the initial data for u_Ψ are in L^2_{loc}. Thus we can represent $u_\Psi(x, t)$ locally using the spectral representation of $\bar{L}$ (as in the proof of Lemma 2.1)

$$u_\Psi = (x, t) = \int \cos\sqrt{\lambda}\, t dE_\lambda u_\Psi(0) + \frac{\sin\sqrt{\lambda}\, t}{\sqrt{\lambda}}\, dE_\lambda(u_\Psi)_t(0).$$

It follows from Lemma 2.1 that if $\phi_\varepsilon \in C_0^\infty(\mathbb{R})$

$$u_\varepsilon(x, t) = \int u_\Psi(x, s)\, \phi_\varepsilon(t-s)\, ds$$

is in C_0^∞. We take ϕ_ε to be an approximate identity. Since $u_\Psi(x, t)$ is in $W^{1,2}_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R})$ $u_{\phi_\varepsilon}(x, t)$ converges to $u_\Psi(x, t)$ in $W^{1,2}_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R})$ as $\varepsilon \downarrow 0$. Moreover, by the Hausdorff–Young inequality:

$$\int |u_\varepsilon(x, t)|^2 dt \leqslant \int |u_\psi(x, t)|^2 dt \tag{2.6}$$

and also

$$\int |u_\Psi(x, t)|^2 dt \leqslant C \int |u(x, t)|^2 dt.$$

Thus, the Lebesgue dominated convergence theorem applies to show

$$\lim_{\varepsilon \downarrow 0} \int q(x) |u_\varepsilon(x, t)|^2 dx\, dt = \int q(x) |u_\Psi^2(x, t)|\, dx\, dt. \tag{2.7}$$

Equation (2.4) applies to the sequence u_ε to give

$$\int_0^{X+T} [u_{\varepsilon_x}^2(0) + q u_\varepsilon^2(0) + u_{\varepsilon_t}^2(0)]$$

$$\leqslant \frac{1}{T} \int_0^T \int_0^{X+T} [u_{\varepsilon_x}^2 + q u_\varepsilon^2 + u_{\varepsilon_t}^2]\, dx\, dt.$$

Letting $\varepsilon \to 0$ we obtain the same inequality for u_Ψ.

Ψ is now taken to be an approximate identity Ψ_ε. u_{Ψ_ε} satisfies (2.4) for each ε. u_{Ψ_ε} converges to u in $W^{1,2}_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R})$. u_{Ψ_ε} satisfies (2.6) and thus

$$\lim_{\varepsilon \to 0} \int\int q(x)\, u^2_{\Psi_\varepsilon}(x, t)\, dx\, dt = \int\int q(x)\, u^2(x, t)\, dx\, dt.$$

Hence we can allow $\varepsilon \to 0$, in (2.4) to obtain (2.4) for u. The initial data of u_{Ψ_ε} converge to the initial data of u and thus the lemma follows. ∎

The first theorem is:

THEOREM 2.1. *Suppose $q(x)$ satisfies* (2.1)(1), (2), *and* (3) *then*

$$u_{tt} - u_{xx} + qu = 0$$

does not have an IO-solution of locally finite energy.

Proof. To prove the theorem we take the Fourier transform of $u(x, t)$ in the t-variable

$$\begin{aligned} \hat{u}(x, k) &= \int_{-\infty}^{\infty} u(x, t)\, e^{-ikt} dt \\ &= \int_{-x}^{x} u(x, t)\, e^{-ikt} dt. \end{aligned}$$

As $u(x, t)$ is assumed to have locally finite energy it follows from (2.2) and the positivity hypotheses on q that for $x < X$

$$\int_0^x u_x^2(x, t)\, dx \leqslant C_X < \infty.$$

The constant depends on X. $u(x, t)$ is an IO-solution and thus $u(0, t) = 0$, therefore

$$\begin{aligned} |u(x, t)| &\leqslant \int_0^x |u_x(s, t)|\, ds, \qquad x < X \\ &\leqslant \sqrt{C_X x}. \end{aligned} \tag{2.8}$$

From this we obtain an estimate on $\hat{u}(x, k)$:

$$\begin{aligned} |\hat{u}(x, k) &\leqslant \int_{-x}^{x} |u(x, t)|\, dt \\ &\leqslant \sqrt{C_X}\, x^{3/2}. \end{aligned} \tag{2.9}$$

From Lemma 2.1 it follows easily that $\hat{u}(x, k)$ is a C^∞ function of x. As $u(x, t)$ satisfies the weak form of (1.1) it follows that

$$\int_0^\infty \int_{-\infty}^\infty u(x, t)(L\phi(x, t) + \phi_{tt}(x, t))\, dx\, dt = 0$$

for $\phi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R})$. The Parseval relation then implies

$$\int_0^\infty \int_{-\infty}^\infty \hat{u}(x, k)(L\hat{\phi}(x, k) - k^2\hat{\phi}(x, k))\, dk\, dx = 0.$$

We can interchange the order of the integrations and integrate by parts to obtain

$$0 = \int_{-\infty}^\infty \int_0^\infty [L\hat{u}(x, k) - k^2\hat{u}(x, k)]\, \hat{\phi}(x, k)\, dx\, dk.$$

The integration by parts is allowable as $\hat{u}(x, k)$ is in C^∞. The set $\{\hat{\phi}(x, k) \colon \phi(x, k) \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R})\}$ is dense in $L^2(\mathbb{R}_+ \times \mathbb{R})$ thus it follows that

$$L\hat{u}(x, k) = k^2 u(x, k) \qquad \text{for} \quad a.e.\ k. \tag{2.10}$$

Equation (2.9) implies that

$$\hat{u}(0, k) = 0$$
$$\hat{u}_x(0, k) = 0.$$

The uniqueness theorem for a solution to (2.9) where q satisfies

$$\int_0^1 x|q(x)|\, dx < \infty \qquad \text{(Coddington and Levinson [1])}$$

implies that $\hat{u}(x, k) = 0$ for a.e. k. The theorem follows as

$$\int_{-\infty}^\infty \int_0^x |u(x, t)|^2 dx\, dt = \int_{-\infty}^\infty \int_0^x |\hat{u}(x, k)|^2 dx\, dk.$$

3. Some Estimates

For a potential to satisfy the non-singularity hypothesis of the previous section it must behave essentially as $O(x^{\varepsilon - 2})$ for some positive ε. In the remainder of the paper we will treat the case of

$$q \sim \alpha/x^2 \qquad \text{as} \quad x \to 0.$$

As we are constructing solutions to the wave equation (1.1) it is necessary to study the eigenfunctions of the operator L. In this section we will obtain estimates for these eigenfunctions.

These eigenfunctions are solutions to integral equations similar to those occurring in the scattering theory of radial potentials. However, the Green's function which we use is not the usual one defined by the operator

$$\frac{-d^2}{dx^2}+\frac{l(l+1)}{x^2}$$

but rather the one defined by

$$L_l=\frac{-d^2}{dx^2}+\frac{l(l+1)}{\sinh^2 x}. \tag{3.1}$$

It is this fact which accounts for the differences in the estimates. Estimates for the eigenfunctions and the Green's function of L_l are presented in an appendix with sketches of their proof.

Let $q(x)$ be a smooth function on $\mathbb{R}^+$ such that for some real $l>\frac{1}{2}$

$$\int_0^1 x\left|q(x)-\frac{l(l+1)}{\sinh^2 x}\right| dx<\infty. \tag{3.2}$$

Let $\bar{q}(x)$ denote $q(x)-l(l-1)/\sinh^2 x$.

Let $\phi(x, \lambda)$ denote the eigenfunction

$$\text{(a)}\quad L\phi(x, \lambda)=\lambda^2\phi(x, \lambda) \tag{3.3}$$

with

$$\text{(b)}\quad \lim_{x\to 0}\frac{\phi(x, \lambda)}{x^{l+1}}=1.$$

If $\bar{q}=0$ we denote this eigenfunction by $P_l(x, \lambda)$. In terms of the hypergeometric function

$$P_l(x, \lambda)=C_l\left(\frac{\operatorname{sh} x}{1+\operatorname{ch} x}\right)^{1/2}\left(\frac{\operatorname{ch} x-1}{\operatorname{ch} x+1}\right)^{(2l+1)/4}(\operatorname{ch} x+1)^{i\lambda}$$
$$\times F\left(\frac{1}{2}-i\lambda, 1-l-i\lambda; \frac{3}{2}+l; \frac{\operatorname{ch} x-1}{\operatorname{ch} x+1}\right).$$

Another linearly independent solution is given by

$$Q_l(x, \lambda)=\frac{D_l\Gamma(l+i\lambda+1)}{\Gamma(1+i\lambda)}\left(\frac{\operatorname{sh} x}{e^x}\right)^{l+1}e^{-i\lambda x}$$
$$\times F(l+1, l+1+i\lambda; 1+i\lambda; e^{-2x}).$$

The condition is satisfied by $Q_l(x, \lambda)$ at zero is

$$\lim_{x \to 0} Q_l(x, \lambda)\, x^l = 1.$$

$P_l(x, \lambda)$ is an entire function of λ for each fixed x and $Q_l(x, \lambda)$ is holomorphic for $\operatorname{Im} \lambda < l + 1$. $\phi(x, \lambda)$ is a solution to the integral equation

$$\phi(x, \lambda) = P_l(x, \lambda) + \int_0^x G_l(x, y, \lambda)\, \bar{q}(y)\, \phi(y, \lambda)\, dy \tag{3.4}$$

where

$$G_l(x, y, \lambda) = P_l(x, \lambda)\, Q_l(y, \lambda) - P_l(y, \lambda)\, Q_l(x, \lambda).$$

If we let K be a constant to be determined later and define

$$W(x) = \left[\exp K \int_0^x |\bar{q}(s)|\, s\, ds\right] \cdot \left[K \int_0^x |\bar{q}(s)|\, s\, dy\right]$$

then the following estimates for $\phi(x, \lambda)$ follow from the estimates for $P_l(x, \lambda)$ and $G_l(x, y, \lambda)$ given in the Appendix.

PROPOSITION 3.1. *$\phi(x, \lambda)$ is an entire function of λ for each fixed x of exponential type x.*

PROPOSITION 3.2. *For λ real:*

(i) $|\phi(x, \lambda)| \leqslant K(1 + W(x))\, x^{l+1}$, $\quad x < \frac{1}{4}, |x\lambda| \leqslant \frac{1}{2}$

(ii) $|\phi(x, \lambda)| \leqslant \dfrac{K(1 + W(x))}{(1 + |\lambda|)^{l+1}}$, $\quad x < \frac{1}{4}, |x\lambda| \geqslant \frac{1}{2}$

(iii) $|\phi(x, \lambda)| \leqslant \dfrac{x}{(1 + |\lambda|\, x)} \dfrac{1}{(1 + |\lambda|)^l}$, $\quad x \geqslant \frac{1}{4}$

(iv) $\left|\dfrac{d\phi(x, \lambda)}{dx}\right| \leqslant K(1 + W(x))\, x^l$, $\quad x < \frac{1}{4}, |x\lambda| < \frac{1}{2}$

(v) $\left|\dfrac{d\phi(x, \lambda)}{dx}\right| \leqslant \dfrac{K(1 + W(x))}{(1 + |\lambda|)^l}$, $\quad x \geqslant \frac{1}{4}$ or

$x < \frac{1}{4}, |x\lambda| \geqslant \frac{1}{2}$.

Proof of Proposition 3.1. $\phi(x, \lambda)$ can also be defined as the solution to the integral equation

$$\phi(x, \lambda) = \frac{j_l(x\lambda)}{\lambda^{l+1}} + \frac{1}{\lambda} \int_0^x H_l(x, y, \lambda)\, \bar{\bar{q}}(y)\, \phi(y, \lambda)\, dy \tag{3.5}$$

where

$$\bar{\bar{q}} = q(x) - l(l+1)/x^2.$$

$j_l(z)$ is the spherical Bessel function

$$j_l(z) = (\pi z/2)^{1/2} J_{l+1/2}(z)$$

and $H_l(x, y, \lambda)$ is the Green's function

$$H_l(x, y, \lambda) = [j_l(\lambda x)\, k_l(\lambda y) - j_l(\lambda y)\, k_l(\lambda x)].$$

$k_l(z)$ is the spherical Bessel functions

$$k_l(z) = (\pi z/2)^{1/2} K_{l+1/2}(z).$$

H_l is analytic in the upper half plane and satisfies for $0 \leqslant y \leqslant x$

$$|H_l(x, y, \lambda) \leqslant K e^{\operatorname{Im} \lambda (x-y)} \frac{(1 + |\lambda y|)^l}{|\lambda y|^l} \left(\frac{|\lambda x|}{1 + |\lambda x|} \right)^{l+1}.$$

$j_l(x\lambda)/\lambda^{l+1}$ is entire in λ and satisfies

$$\frac{|j_l(\lambda x)|}{\lambda^{l+1}} \leqslant \frac{K e^{|\operatorname{Im} \lambda|\, x} |x|^{l+1}}{(1 + |\lambda x|)^{l+1}}.$$

These estimates are standard and can be found in [3].

As $\phi(x, \lambda)$ is the solution to (3.5) it is the limit of a locally uniformly convergent sequence of functions analytic in the upper half plane and is thus analytic in the upper half plane. Moreover $\phi(x, \lambda)$ is real for λ real and therefore we can continue $\phi(x, \lambda)$ to the lower half plane by

$$\phi(x, \bar{\lambda}) = \overline{\phi(x, \lambda)}.$$

From standard estimates [3] it follows that $\phi(x, \lambda)$ is of exponential type x. As L has real coefficients it is clear that

$$L\phi(x, \bar{\lambda}) = \bar{\lambda}^2 \phi(x, \bar{\lambda}).$$

This completes the proof of Proposition 3.1. ∎

Proof of Proposition 3.2. To prove the estimates listed in the

Proposition it is necessary to use (3.4). Estimates on P_l and G_l can be found in the Appendix:

Proof of (i). $x<\frac{1}{4}$, $|x\lambda|<\frac{1}{2}$.

$$|\phi(x,\lambda)|\leqslant C\left[x^{l+1}+\int_0^x \frac{x^{l+1}}{y^l}|\bar{q}(y)\,\phi(y,\lambda)|\,dy\right]$$

Thus

$$\left|\frac{\phi(x,\lambda)}{x^{l+1}}\right|\leqslant C\left[1+\int_0^x y|\bar{q}(y)|\left|\frac{\phi(y,\lambda)}{y^{l+1}}\right|dy\right].$$

A standard application of Gronwall's inequality implies

$$|\phi(x,\lambda)|\leqslant x^{l+1}[1+W(x)].$$

In (ii), $x<\frac{1}{4}$, $|x\lambda|\geqslant 1$,

$$|\phi(x,\lambda)|\leqslant\frac{K}{(1+|\lambda|)^{l+1}}+\int_0^{(1/2)|\lambda|}\frac{K}{(1+|\lambda|)^{l+1}}|\bar{q}(y)|\frac{|\phi(y,\lambda)|}{y^l}\,dy$$
$$+\int_{(1/2)|\lambda|}^x\frac{K(1+|\lambda|)^l}{(1-|\lambda|)^{l+1}}|\phi(y,\lambda)\,\bar{q}(y)|\,dy.$$

Using part (i) in the first integral we obtain

$$|\phi(x,\lambda)|\leqslant\frac{K}{(1+|\lambda|)^{l+1}}+\int_{(1/2)|\lambda|}^x Ky|\bar{q}(y)|\,|\phi(y,\lambda)|dy.$$

Again a standard application of the Gronwall inequality implies that

$$|\phi(x,\lambda)|\leqslant\frac{K(1+W(x))}{(1+|\lambda|)^{l+1}}.$$

In (iii), $\frac{1}{4}<x$, we break the integral into three parts and use (i) and (ii):

$$|\phi(x,\lambda)|\leqslant\frac{Kx}{(1+|\lambda|\,x)}\frac{1}{(1+|\lambda|)^l}$$
$$+\int_0^{(1/2)|\lambda|}\frac{x}{(1+|\lambda|\,x)}\frac{y}{(1+|\lambda|)^l}|\bar{q}(y)|\,dy$$
$$+\int_{(1/2)|\lambda|}^{1/4}\frac{Kx}{(1+|\lambda|\,x)}\frac{1}{(1+|\lambda|)^l}(1+W(y))\,y|\bar{q}(y)|\,dy$$
$$+\int_{1/4}^x\frac{x}{(1+|\lambda|x)}\cdot\frac{(1+|\lambda|)^l}{(1+|\lambda|)^l}|\bar{q}(y)\,\phi(y,\lambda)|\,dy.$$

Thus

$$\frac{(1+|\lambda|)^l(1+|\lambda|\,x)}{x}\,|\phi(x,\lambda)| \leqslant K\left[1+\int_{1/4}^{x} y|\bar{q}(y)|(1+|\lambda|)^l\frac{|\phi(y,\lambda)|}{y}\,dy\right]$$

$$\leqslant K\left[1+\int_{1/4}^{x} y\bar{q}(y)(1+|\lambda|)^l(1+|\lambda|\,y)\frac{|\phi(y,\lambda)|}{y}\,dy\right]$$

from which

$$|\phi(x,\lambda)| \leqslant K(1+W(x))\frac{x}{(1+|\lambda|\,x)}\frac{1}{(1+|\lambda|)^l}$$

follows immediately. The estimates for the derivatives are done in a very similar way, and we will omit the proofs.

4. Incoming and Outgoing Waves

In this section we will construct IO-solutions for (1.1) of locally finite energy if for some $l>\frac{1}{2}$

$$\int_0^1 x\left|q(x)-\frac{l(l+1)}{\sinh^2 x}\right|dx<\infty \tag{4.1}$$

and IO-solutions of finite energy if for some $l>\frac{1}{2}$

$$\int_0^\infty x\left|q(x)-\frac{l(l+1)}{\sinh^2 x}\right|dx<\infty. \tag{4.2}$$

We will construct $d_l=[\![l+\frac{1}{2}-\varepsilon]\!]$ ($[\![\cdot]\!]$ greatest integer, $\varepsilon>0$ arbitrarily small) linearly independent solutions in each case. For l not a half integer these solutions are probably a basis for the space of all such solutions.

After the proofs of Theorems 4.1 and 4.2 we will comment again on this matter.

THEOREM 4.1. *If q satisfies* (4.1) *then there is a $d_l=[\![l+\frac{1}{2}-\varepsilon]\!]$ (for every $\varepsilon>0$) dimensional space of IO-solutions with locally finite energy.*

THEOREM 4.2. *If q satisfies* (4.2) *then these solutions have finite total energy.*

Suppose $p(\lambda)$ is a polynomial of degree at most d_l-1 then it follows easily from Propositions 3.1 and 3.2 that $p(\lambda)\,\phi(x,\lambda)$ is an entire function of exponential type x which is in $L^2(\mathbb{R})$ hence the Paley–Wiener theorem implies that for each x

$$u(x, t) = \int_{-\infty}^{\infty} e^{-i\lambda t}\phi(x, \lambda)\, p(\lambda)\, d\lambda$$

is in $L^2(\mathbb{R})$ and furthermore

$$u(x, t) = 0 \qquad \text{if} \quad x \leqslant |t|.$$

Thus $u(x, t)$ is an IO-function; it is also easily seen to be a weak solution to the wave equation (1.1). To complete the proof all we need are energy estimates. Observe that if $u(x, t)$ is a smooth solution to the wave equation then

$$\int_{-T}^{T}\int_{\varepsilon}^{X} (u_x^2 + u_t^2 + qu^2)\, dx\, dt$$
$$= \int_{-T}^{T} [u_x(X, t)\, u(X, t) - u_x(\varepsilon, t)\, u(\varepsilon, t)]\, dt$$
$$+ \int_{-T}^{T}\int_{\varepsilon}^{X} [(uLu) + u_t^2]\, dx\, dt. \tag{4.3}$$

We define $u^m(x, t) = \int_{-m}^{m} e^{-i\lambda t} p(\lambda)\, \phi(x, \lambda)\, d\lambda$ and moreover specialize to $p(\lambda) = \lambda^k$ for $k \leqslant d_l - 1$.

We apply (4.3) to $u^m(x, t)$ and apply the Parseval relation to obtain for $T > X$

$$\int_{-T}^{T}\int_{\varepsilon}^{X} [(u_x^m)^2 + (u_t^m)^2 + (u^m)^2 q]\, dx\, dt$$
$$\leqslant \int_{-m}^{m} [\lambda^{2k}\phi(X, \lambda)\, \phi_x(X, \lambda) - \lambda^{2k}\phi(\varepsilon, \lambda)\, \phi_x(\varepsilon, \lambda)]\, d\lambda$$
$$+ \int_{-m}^{m}\int_{\varepsilon}^{X} \lambda^{2+2k}|\phi(x, \lambda)|^2 dx\, d\lambda.$$

We estimate each term:

$$\int_{-m}^{m} \lambda^{2k}|\phi(X, \lambda)\, \phi_x(X, \lambda)|\, d\lambda \leqslant KX(1 + W(X))^2 \int_0^{\infty} \frac{\lambda^{2k} d\lambda}{(1 + |\lambda|)^{2l+1}}$$
$$\leqslant KX(1 + W(X))^2 \qquad \text{if } k \leqslant d_l - 1$$

$$\int_{-m}^{m} \lambda^{2k}|\phi(\varepsilon, \lambda)\, \phi_x(\varepsilon, \lambda)|\, d\lambda \leqslant K\int_0^{1/\varepsilon} \lambda^{2k}\varepsilon^{2l+1} d\lambda + K\int_{1/\varepsilon}^{\infty} \frac{\lambda^{2k}\varepsilon\, d\lambda}{(1 + |\lambda|)^{2l}}$$
$$\leqslant K\varepsilon^{2(l-k)} \qquad \text{if} \quad k \leqslant d_l - 1$$

$$\int_{-m}^{m}\int_{\varepsilon}^{X} \lambda^{2+2k}|\phi(x,\lambda)|^2 dx\,d\lambda \leqslant \Bigg[\int_{\varepsilon}^{1/4}\int_{0}^{2/x} \lambda^{2+2k}x^{2l+2}d\lambda\,dx$$

$$+\int_{\varepsilon}^{1/4}\int_{2/x}^{\infty}\frac{\lambda^{2+2k}}{(1+|\lambda|)^{2l+2}}\,d\lambda\,dx$$

$$+\int_{0}^{\infty}\int_{1/4}^{X}\frac{\lambda^{2+2k}x^2dx\,d\lambda}{(1+|\lambda|\,x)^2(1+|\lambda|)^{2l}}\Bigg](1+W(X))^2$$

$$\leqslant\left[C+x\int_{0}^{\infty}\frac{\lambda^{2k}d\lambda}{(1+|\lambda|)^{2l}(1+W(X))^2}\right]$$

$$\leqslant C(1+X)(1+W(X))^2 \qquad \text{if} \quad k\leqslant d_l-1.$$

Thus we've obtained the estimate

$$\int_{\infty}^{\infty}\int_{\varepsilon}^{X}[(u_x^m)^2+(u_t^m)^2+(u^m)^2q]\,dx\,dt\leqslant C[(1+X)(1+W(X))^2+\varepsilon^{2(l-k)}]$$

Clearly we can allow $\varepsilon\to 0$ and $m\to\infty$ to obtain

$$\int_{\infty}^{\infty}\int_{0}^{X}(u_x^2+u_t^2+qu^2)\,dx\,dt\leqslant C(1+X)(1+W(X))^2.$$

It follows from Lemma 2.2 that $u(x,t)$ is a solution to the wave equation with locally finite energy. This completes the proof of Theorem 4.1.

The estimate in Lemma 2.2 states

$$\int_{0}^{X}(u_x^2(x,0)+u_t^2(x,0)+qu^2(x,0))\,dx\leqslant\frac{C}{X}(1+2X)(1+W(X))^2. \tag{4.4}$$

If $\lim_{x\to\infty} W(x)<\infty$ then we can allow $x\to\infty$ in (4.4) to obtain

$$\int_{0}^{\infty}[u_x(x,0)+u_t^2(x,0)+qu^2(x,0)]\,dx\leqslant K(1+W(\infty))^2$$

thus completing the proof of Theorem 4.2.

5. Applications to Wave Equations in Space

Theorems 4.1 and 4.2 can be used to construct incoming and outgoing solutions to the wave equation

$$u_{tt}=(\Delta-q(r))\,u$$

on either hyperbolic or Euclidean $n+1$-space. This is accomplished by considering solutions of the form

$$V(r, t)\, Y_m(\theta)$$

where $Y_m(\theta)$ is a spherical harmonic that satisfies

$$\Delta_{S^n} Y_m = -m(m+n-1)\, Y_m, \qquad m \in \mathbb{N}.$$

Under suitable hypotheses on $q(r)$ we will show that in hyperbolic space there are IO-solutions of finite total energy while in Euclidean space there are not. In this context a solution $u(x, t)$ is IO if

$$u(x, t) = 0 \qquad \text{for} \quad |x| < |t|.$$

First we consider hyperbolic space. In geodesic normal coordinates the Laplace operator is

$$\Delta_{\mathbb{H}^{n+1}} = \frac{1}{\sinh^n r}\, \partial_r \sinh^n r\, \partial_r + \frac{1}{\sinh^n r}\, \Delta_{S^n}.$$

The wave equation is

$$u_{tt} = (\Delta_{\mathbb{H}^{n+1}} + (n/2)^2 - q(r))\, u.$$

Suppose $u(r, \theta, t)$ is a solution of the form

$$u(r, \theta, t) = \frac{V(r, t)\, Y_m(\theta)}{\sinh^{n/2} r}$$

and let

$$l_m = m + (n-2)/2$$

then $V(r, t)$ satisfies

$$V_{tt} = V_{rr} - \left\{\frac{l_m(l_m+1)}{\sinh^2 r} + q(r)\right\} V, \tag{5.1}$$

and the energy of u is

$$E(r) = \int_0^\infty \left[V_r^2 + \left[\frac{l_m(l_m+1)}{\sinh^2 r} + q(r)\right] V^2 + V_t^2 \right] dr.$$

with these observations the following theorem is an immediate consequence of Theorem 4.2:

THEOREM 5.1. *If $q(r)$ satisfies the hypotheses of Theorem 4.2 then the wave equation*

$$u_{tt} = (\Delta_{\mathbb{H}^{n+1}} - (n/2)^2 - q(r))\, u \tag{5.2}$$

has for each $m \in \mathbb{N}$ *an* $[\![l_m]\!]$*-dimensional space of initial data of the form* $(V_1(r)\, Y_m(\theta),\ V_2(r)\, Y_m(\theta))$ *such that the solution of* (5.2) *with this data finite energy and is incoming and outgoing.*

Remark. The arguments in Theorem 4.2 do not include the case $n=1$, $m=1$; this requires a special argument which we omit.

Lax and Phillips proved the existence of such solutions when $q \equiv 0$. The dimension of their space of solutions agrees with ours. For more on perturbations of the wave equation on $\mathbb{H}^{n+1}$ see [6, 5].

The theorem in Euclidean space requires a bit more work as we have to identify all possible finite energy IO-solutions and prove that the energy diverges.

In $\mathbb{R}^{n+1}$ we seek solutions to

$$u_{tt} = (\Delta_{\mathbb{R}^{n+1}} - q(r))\, u \tag{5.3}$$

which are of locally finite energy and of the form

$$u(r, \theta, t) = \frac{V(r, t)}{r^n}\, Y_m(\theta).$$

V satisfies

$$V_{tt} = V_{rr} - \left(\frac{l_m(l_m+1)}{r^2} + q(r)\right) V, \tag{5.4}$$

and the local energy of u is

$$E_R(V) = \int_0^R \left(V_r^2 + \left(\frac{l_m(l_m+1)}{r^2} + q(r)\right) V^2 + V_t^2\right) dr.$$

We will suppose that $q(r) \geqslant 0$ and that

$$\int_0^\infty r^2 |q(r)|\, dr < \infty. \tag{5.5}$$

We assume (5.5) so that we can use certain asymptotic results well known in scattering theory [7]. The theorem is:

THEOREM 5.2. *If q satisfies the above hypotheses then the equation*

$$u_{tt} = (\Delta_{\mathbb{R}^{n+1}} - q(r))\, u$$

has no IO-solutions of finite total energy.

Proof. Suppose there were such solutions. Then (5.4) would also have such a solution for some $m \geqslant 1$. The theorem in Section 2 states there are no such solutions if $l_m = 0$. Let $V(r, t)$ be a solution to (5.5) with finite energy. That is, $\lim_{R \to \infty} E_R(V) < \infty$. Let $\hat{V}(r, \lambda)$ be the Fourier transform of $V(r, t)$ in t. Using arguments similar to those in Sections 1 and 2 we see that:

(1) $\hat{V}(r, \lambda)$ is smooth in (r, λ),

(2) $\hat{V}(r, \lambda)$ is an entire function of λ of exponential type r,

(3) $\lim_{r \to \infty} \hat{V}(r, \lambda) = 0$,

(4) $\hat{V}_{rr}(r, \lambda) - (l_m(l_m+1)/r^2 + q(r))\, \hat{V} = -\lambda^2 \hat{V}$.

From these facts we conclude that

$$\hat{V}(r, \lambda) = p(\lambda)\, \phi(r, \lambda).$$

$\phi(r, \lambda)$ is the holomorphic family of eigenfunction of $-d^2/dr^2 + l_m(l_m+1)/r^2 + q(r)$ constructed in Section 3 and $p(\lambda)$ is a meromorphic function. Since $\phi(r, \lambda)$ solves the integral equation, (3.5), it is clear that $\phi(r, \lambda_0) \not\equiv 0$ for any λ_0. Thus we see that $p(\lambda)$ is holomorphic. $\phi(r, \lambda)$ is of exponential type exactly r and thus $p(\lambda)$ must be of exponential type zero. In fact $p(\lambda)$ is a polynomial. This follows by using the following well-known fact from scattering theory:

LEMMA 5.1. *If $q(r)$ satisfies (5.5) and $q \geqslant 0$ then for large (λ)*

$$\phi(r, \lambda) = \frac{A(\lambda) \sin(\lambda r - \pi l_m/2 + \psi(\lambda))}{\lambda^{l_m+1}} + O\left(\frac{1}{|\lambda|^{l_m+2}}\right),$$

$$A(\lambda) \sim 1, \quad \psi(\lambda) \sim 0 \qquad \text{(Levinson [3])}.$$

We use this as follows:

Inequality (2.3) states

$$E_{2R}(V) \geqslant \frac{1}{2R} \int_{-R}^{R} \int_0^R \left[V_r^2(r, t) + \left(q(r) + \frac{l_m(l_m+1)}{r^2} \right) V^2 + V_t^2 \right] dr\, dt. \tag{5.6}$$

$V(r, t)$ is supported for $t \in [-r, r]$ hence the Parseval theorem for the Fourier transform implies that (5.6) is equivalent to

$$\begin{aligned} E_{2R}(V) &\geqslant \frac{1}{2R} \int_0^R \int_{-\infty}^{\infty} \left[\hat{V}_r^2 + \left(q + \frac{l_m(l_m+1)}{r^2} \right) \hat{V}^2 + \lambda^2 \hat{V}^2 \right] d\lambda\, dr \\ &\geqslant \frac{1}{2R} \int_0^R \int_{-\infty}^{\infty} \lambda^2 |p(\lambda)|^2 |\phi(r, \lambda)|^2 d\lambda\, dr \\ &\geqslant \frac{c}{2R} \int_{-\infty}^{\infty} \frac{\lambda^2 |p(\lambda)|^2 d\lambda}{(1+|\lambda|)^{2(l_m+1)}}. \end{aligned} \tag{5.6'}$$

We used Lemma 5.1 to obtain a lower bound for

$$\int_0^R |\phi(r, \lambda)|^2 dr.$$

To complete the proof that $p(\lambda)$ is a polynomial we use a Phragmen–Lindelöf-type lemma:

LEMMA 5.2. *Suppose $f(z)$ is an entire function of exponential type zero. Suppose that for some $z_0 \in \mathbb{C}$, $\theta \in (0, 2\pi)$, and $0 \leqslant k < \infty$*

$$\int_{-\infty}^{\infty} \frac{|f(z_0 + re^{i\theta})|^2}{(1+|r|)^{2k}} dr < \infty$$

then f is a polynomial of degree less than $k - 1/2$.

Proof. Without loss of generality we may suppose that $z_0 = 0$ and $\theta = 0$. Let $k' = [\![k]\!]$. Let $f_k(z)$ be the k' degree Taylor polynomial for f at zero. Define

$$F(z) = \frac{f(z) - f_k(z)}{z^{k'+1}}.$$

$F(z)$ is an entire function of exponential type zero. Furthermore

$$\int_{-\infty}^{\infty} |F(x)|^2 dx < \infty$$

thus the Paley–Wiener theorem implies that $F(z)$ is the Fourier transform of an L^2 function supported at zero. This proves the lemma.

From Lemma 5.2 and (5.6′) it is immediate that $p(\lambda)$ is a polynomial of degree at most $[\![l_m]\!] - 1$. To complete the proof we use a standard comparison between $\phi(r, \lambda)$ and $j_{l_m}(r\lambda)/\lambda^{l+1}$.

$$\frac{j_{l_m}(r\lambda)}{\lambda^{l_m+1}} \quad \text{solves} \quad -f'' + \frac{l_m(l_m+1)}{r^2} f = \lambda^2 f$$

and $\lim_{r \to 0} (j_{l_m}(r\lambda)/(r\lambda)^{l_m+1} = 1$. As $q(r) \geqslant 0$ it follows easily from standard comparison theorems that

$$\phi(r, \lambda) \geqslant \frac{j_{l_m}(r\lambda)}{\lambda^{l_m+1}} \tag{5.7}$$

so long as $r\lambda$ is less than the first zero of $j_{l_m}(x)$. This will be so for

$$r\lambda \leqslant l_m.$$

Thus (5.7) holds for $\lambda \in [0, l_m/r]$.

Integrating by parts in (5.6) we obtain

$$\begin{aligned}
2RE_{2R}(V) &\geqslant \int_{-R}^{R}\int_{0}^{R}\left[V\left(-V_{rr}+\left(q+\frac{l_m(l_m+1)}{r^2}\right)V\right)+V_t^2\right] \\
&\quad + \int_{-R}^{R} V_r(R,t)\,V(R,t)\,dt \\
&= \int_0^R\int_{-\infty}^{\infty} 2\lambda^2|\hat{V}(r,\lambda)|^2 dr\,d\lambda \\
&\quad + \frac{d}{dr}\int_{-\infty}^{\infty} |\hat{V}(r,\lambda)|^2 d\lambda\,|_{r=R} \\
&\geqslant \frac{d}{dr}\int_{-\infty}^{\infty} |\hat{V}(r,\lambda)|^2 d\lambda\,|_{r=R}. \qquad (5.8)
\end{aligned}$$

We integrate (5.8) with respect to R to obtain

$$\int_0^R 2r\,E_{2r}(V)\,dr \geqslant \int_{-\infty}^{\infty} |\hat{V}(R,\lambda)|^2 d\lambda. \qquad (5.9)$$

Using (5.7) with $p(\lambda)=\lambda^k$, $k \leqslant [\![l_m]\!]-1$, and the comparison argument it follows that

$$\begin{aligned}
\int_0^R 2r\,E_{2r}(V)\,dr &\geqslant \int_0^{l_m/R} \lambda^{2(k-l_m-1)}|j_{l_m}(\lambda R)|^2 d\lambda \\
&\geqslant CR^{2(l_m-k)+1} \\
&\geqslant CR^3.
\end{aligned}$$

On the other hand if $E_\infty(V)<\infty$ then

$$\int_0^R 2r\,E_{2r}(V)\,dr \leqslant E_\infty R^2.$$

Thus we have derived the necessary contradiction to prove Theorem 5.2.

Remark. The potential need not be strictly positive for Theorem 5.2 to be true. However, if $q(r)$ is such that the solution to

$$-\phi'' + (l(l+1)/r^2+q)\,\phi = 0$$

behaves as $1/r^l$ as $r \to \infty$ it seems possible that $\int e^{i\lambda t}\phi(r,\lambda)\,d\lambda$ have total finite energy.

APPENDIX

In this appendix we derive estimates for the eigenfunctions $P_l(x, \lambda)$, $Q_l(x, \lambda)$ and the Green's function

$$G_l(x, y, \lambda) = P_l(x, \lambda)\, Q_l(y, \lambda) - P_l(y, \lambda)\, Q_l(x, \lambda).$$

The estimates will be for real values of λ. $P_l(x, \lambda)$ is defined by

$$P_l(x, \lambda) = \left[\frac{(\operatorname{sh} x)}{(1+\operatorname{ch} x)}\right]^{1/2} \left(\frac{\operatorname{ch} x - 1}{\operatorname{ch} x + 1}\right)^{(2l+1)/4} (\operatorname{ch} x + 1)^{i\lambda}\, \Gamma(l + 3/2)$$
$$\times F\left(\frac{1}{2} - i\lambda, l + 1 - i\lambda; 3/2 + l; \frac{\operatorname{ch} x - 1}{\operatorname{ch} x + 1}\right)$$

(Gradshteyn and Ryzhik [4]). (A.1)

We will use (A.1) to estimate P_l for small x: The hypergeometric factor equals

$$1 + \sum_{k=0}^{\infty} \frac{(\frac{1}{2} - i\lambda) \cdots (k + \frac{1}{2} - i\lambda)(l + 1 - i\lambda) \cdots (l + 1 + k - i\lambda)}{1 \cdot 2 \cdots k \cdot 1 \qquad (\frac{3}{2} + l) \cdots (\frac{3}{2} + k + l)} \left(\frac{\operatorname{ch} x - 1}{\operatorname{ch} x + 1}\right)^{k+1}$$
$$= 1 + \sum_{k=0}^{\infty} a_k(l, \lambda) \left(\frac{\operatorname{ch} x - 1}{\operatorname{ch} x + 1}\right)^{k+1}.$$

We need to estimate the coefficients

$$a_k(l, \lambda) = \prod_{j=1}^{k} \left(1 - \frac{\frac{1}{2} + i\lambda}{j}\right) \prod_{j=1}^{k} \left(1 - \frac{\frac{1}{2} + i\lambda}{\frac{3}{2} + j + l}\right).$$

Thus

$$|a_k(l, \lambda)| \leqslant \prod_{j=1}^{k} \left(1 + \frac{|\lambda|}{j}\right) \sum_{j=1}^{k} \left(1 + \frac{|\lambda|}{\frac{3}{2} + j + l}\right)$$
$$\begin{cases} \leqslant |\lambda|^{2k} \prod_{j=1}^{k} \left(\frac{1}{|\lambda|} + \frac{1}{j}\right)\left(\frac{1}{|\lambda|} + \frac{1}{\frac{2}{3} + j + l}\right) & |\lambda| \geqslant 1 \\ \leqslant 2(\frac{3}{2})^{2k}, & |\lambda| \leqslant 1 \end{cases}$$
$$\begin{cases} \leqslant 2(\frac{3}{2}|\lambda|)^{2k} & |\lambda| \geqslant 1 \\ \leqslant 2(\frac{3}{2})^{2k}, & |\lambda| \leqslant 1. \end{cases}$$

From these estimates we obtain

$$P_l(x, \lambda) \leqslant K x^{l+1} \qquad \text{for} \quad |x\lambda| \leqslant \tfrac{1}{2},\ x \leqslant \tfrac{1}{4}. \tag{A.2}$$

To obtain other estimates we use a different representation:

$$P_l(x,\lambda)=2^{l+1}\left(\frac{\operatorname{sh}x}{ex}\right)^{l+1}\times\left[\frac{\Gamma(-i\lambda)\,e^{-i\lambda x}}{\Gamma(l+1-i\lambda)}F(l+1+i\lambda,l+1;1+i\lambda;e^{-2x})\right.$$
$$\left.+\frac{\Gamma(i\lambda)\,e^{-i\lambda x}}{\lambda(l+1+i\lambda)}F(l-i\lambda+1,l+1;1-i\lambda;e^{-2x})\right]$$

(Gradshteyn and Ryzhik [4]). (A.3)

$Q_l(x,\lambda)$ has a similar form:

$$Q_l(x,\lambda)=\frac{C_l\Gamma(l+i\lambda+1)}{\Gamma(i\lambda+1)}\left(\frac{\operatorname{sh}x}{e^x}\right)^{l+1}e^{-i\lambda x}$$
$$\times F(l+i\lambda+1,l+1;i\lambda+1;e^{-2x})$$

(Gradshteyn and Ryzhik [4]).

First we will estimate the hypergeometric factor:

$$F(l+1+\lambda,l+1;i\lambda+1;e^{-2x})$$
$$=1+\sum\frac{(l+1+i\lambda)\cdots(l+h+i\lambda)}{(i\lambda+1)\cdots(k+i\lambda)}\cdot\frac{(l+1)\cdots(l+k)}{1...k}e^{-2xk}$$
$$=1+\sum_{k+1}^{\infty}b_k(l,\lambda)\,e^{-2xk}$$
$$b_k(l,\lambda)=\left(1+\frac{l}{i\lambda+1}\right)\cdots\left(1+\frac{l}{i\lambda+k}\right)\cdot(l+1)\cdots\left(1+\frac{l}{k}\right).$$

Now we estimate $b_k(l,\lambda)$:

$$\log|b_k(l,\lambda)|\leqslant\sum_{j=1}^{k}\log\left(1+\frac{l}{\sqrt{j^2+\lambda^2}}\right)+\sum_{j=1}^{k}\log\left(1+\frac{l}{j}\right)$$
$$\begin{cases}\leqslant[2+\log k+\log(k/\lambda+\sqrt{1+(k/\lambda)^2})]\,l, & |\lambda|\geqslant 1\\ \leqslant[2+2\log(k+1)]\,l, & |\lambda|<1.\end{cases}$$

Thus

$$b_k(l,\lambda)\begin{cases}\leqslant Kk^l, & k\leqslant|\lambda|\\ \leqslant Kk^{2l}, & |\lambda|\leqslant 1\\ \leqslant\dfrac{Kk^{2l}}{|\lambda|^l}, & k\geqslant|\lambda|.\end{cases}$$

Thus

$$F(l+i\lambda+1, l+1; 1+i\lambda; e^{-2x}) \begin{cases} \leqslant 1+\dfrac{K}{x^{l+1}}\left(1+\dfrac{1}{|x\lambda|^{l}}\right), & \\ \quad\text{or} & |\lambda|\geqslant 1 \\ \leqslant 1+K\left[|\lambda|^{l+1}+\dfrac{1}{x^{2l+1}|\lambda|^{l}}\right], & \\ \leqslant 1+\dfrac{K}{x^{2l+1}}, \qquad |\lambda|<1 & \end{cases} \tag{A.4}$$

and therefore

$$\begin{aligned} &\text{(a)}\quad |P_l(x,\lambda)|\leqslant Kx^{l+1}, && x<\tfrac{1}{4};\ |x\lambda|\leqslant\tfrac{1}{2} \\ &\text{(b)}\quad |P_l(x,\lambda)|\leqslant\frac{K}{(1+|\lambda|)^{l+1}}, && x<\tfrac{1}{4};\ |x\lambda|\geqslant\tfrac{1}{2} \\ &\text{(c)}\quad |P_l(x,\lambda)|\leqslant\frac{Kx}{(1+|\lambda|\,x)}\frac{1}{(1+|\lambda|)^{l}}, && x\geqslant\tfrac{1}{4} \end{aligned} \tag{A.5}$$

$$\begin{aligned} &\text{(a)}\quad |Q_l(x,\lambda)|\leqslant\frac{K}{x^{l}}, && |\lambda x|\leqslant\tfrac{1}{2};\ x<\tfrac{1}{4} \\ &\text{(b)}\quad |Q_l(x,\lambda)|\leqslant K(1+|\lambda|)^{l}, && |x\lambda|\geqslant\tfrac{1}{2};\ x\leqslant\tfrac{1}{4}\text{ or }x\geqslant\tfrac{1}{4} \end{aligned} \tag{A.6}$$

$$\frac{dF}{dx}(l+i\lambda+1, l+1; 1+i\lambda; e^{-2x}) \begin{cases} \leqslant\dfrac{K}{x^{l+2}}\left(1+\dfrac{1}{(x|\lambda|)^{l}}\right), & |\lambda|\geqslant 1 \\ \leqslant\dfrac{K}{x^{2l+2}}, & |\lambda|<1 \end{cases} \tag{A.7}$$

$$\begin{aligned} &\text{(a)}\quad \left|\frac{dP_l(x,\lambda)}{dx}\right|\leqslant Kx^{l}, && |x\lambda|\leqslant\tfrac{1}{2},\ x\leqslant\tfrac{1}{4} \\ &\qquad\ \left|\frac{dP_l(x,\lambda)}{dx}\right|\leqslant\frac{K}{(1+|\lambda|)^{l}}, && |x\lambda|\geqslant\tfrac{1}{2},\ x<\tfrac{1}{4} \\ &\qquad\ \left|\frac{dP_l(x,\lambda)}{dx}\right|\leqslant\frac{K}{(1+|\lambda|)^{l}}, && x\geqslant\tfrac{1}{4} \end{aligned} \tag{A.8}$$

and analogous estimates for dQ_l/dx.

Finally, we combine these estimates to estimate G_l:

$$\begin{aligned}
&\text{(a)} \quad |G_l(x, y, \lambda)| \leqslant K \frac{x^{l+1}}{y^l}, && 0 < y < x < \frac{1}{4}, |x\lambda| \leqslant \frac{1}{2} \\
&\text{(b)} \quad |G_l(x, y, \lambda)| \leqslant \frac{K}{(1+|\lambda|)^{l+1} y^l}, && 0 < y < x < \frac{1}{4}, |y\lambda| < \frac{1}{2} < |x\lambda| \\
&\text{(c)} \quad |G_l(x, y, \lambda)| \leqslant K \frac{(1+|\lambda|)^l}{(1+|\lambda|)^{l+1}} && 0 < y < x < \frac{1}{4}, \frac{1}{2} < |y\lambda| \\
&\text{(d)} \quad |G_l(x, y, \lambda)| \leqslant \frac{K}{y^l} \frac{x}{(1+|\lambda|\, x)} \frac{1}{(1+|\lambda|)^l}, && 0 < y < \frac{1}{4} < x, |y\lambda| < \frac{1}{2} \\
&\text{(e)} \quad |G_l(x, y, \lambda)| \leqslant \frac{Kx}{(1+|\lambda|\, x)} \frac{(1+|\lambda|)^l}{(1+|\lambda|)^l}, && 0 < y < \frac{1}{4} < x, |y\lambda| \geqslant \frac{1}{2} \\
&\text{(f)} \quad |G_l(x, y, \lambda)| \leqslant \frac{Kx}{(1+|\lambda|\, x)} \frac{(1+|\lambda|)^l}{(1+|\lambda|)^l}, && \frac{1}{4} < y < x.
\end{aligned} \tag{A.9}$$

Using the bounds for the derivatives of the eigenfunctions one can derive bounds for $|(dG_l/dx)(x, y, \lambda)|$ analogous to (a)–(f).

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